mode was obtained, the arc voltage abruptly dropped slightly in magnitude and achieved voltage stability. As the arc current was increased, the voltage increased and the arc chamber pressure either decreased slightly or remained constant. The pressure behavior was a definite indication of coupling of the energy supplied to the MHD arc to the ionization and excitation of the Ar plasma. With loss of the desired arc mode, indicated by a sudden drop in arc voltage and increase in pressure, generation of the laser states in the supersonic expansion ceased.

The magnitude of the arc current required for excitation of the electronic states depended on the electronic state. At an MHD arc chamber pressure of 50 torr, the threshold current required for generation of the $4s^2P_{1/2}$ energy level in the supersonic expansion was 340–380 amp, whereas the threshold current for the $4s^2P_{3/2}$ energy level was 420–500 amp.

The absorption measurements indicate positive existence only of the $4s^2P_{1/2}$ and $4s^2P_{3/2}$ lower laser levels of the Ar^+-ion. However, because of the existence in the supersonic expansion of these lower laser levels, the upper laser levels $4p^2P^0_{3/2}$, $4p^2D^0_{5/2}$, and $4p^4D^0_{5/2}$ of the 4765, 4880, and 5145Å transitions also are expected to be present. In the two-step excitation process by electron impact through the ground state $3s^23p^5\ ^2P^0$ of the singly charged Ar^+-ion, one would expect, based on parity considerations, that excitation to the lower 4*s* levels should be more favorable than excitation to the upper 4*p* levels.[2] However, the theoretical calculations of Beigman et al.[9] suggest that the total excitation cross sections for the 4*p* and the 4*s* levels may well be of the same magnitude. In fact, according to their numerical results, excitation to the 4*p* configuration is favoured at high electron temperature. Through appropriate choice of the independent variables affecting the excitation parameters, one thus should be able to directly obtain population inversion of the Ar^+-ion states in the supersonic expansion.

Through variation of the Ar mass flow rate, sustained laser gain at 5145Å was obtained. Figure 1 is a typical representation of the obtained laser gain. The stagnation pressure was first increased to 130 torr, resulting in laser absorption. The arc voltage is very noisy until the arc current is increased. With increase in arc current laser gain commences and continues over a wide range of arc current. For the conditions of Fig. 1 a laser gain of 1.7% per meter was achieved.

The laser gain is most sensitive to the stagnation pressure and mass flow rate of Ar. The only essential differences in the conditions for which absorption occurred and for which gain occurred are the stagnation pressure and the mass flow rate of Ar. For example, with the same arc current, absorption would occur at a mass flow rate of 3.8 g/sec while gain would occur at a mass flow rate of 4.8 g/sec. With either increase or decrease of the mass flow rate from 4.8 the gain in Fig. 1 disappeared. This critical dependence of the laser gain on the mass flow rate is being investigated further.

The statement by Smith et al. for the observed delayed threshold laser oscillation for a pulsed static discharge tube applies equally well for the supersonic expansion, "It is difficult to envisage such excitation processes, cascades from the more highly ionized Ar states, or any plasma effect providing lifetimes as long as this."[5]

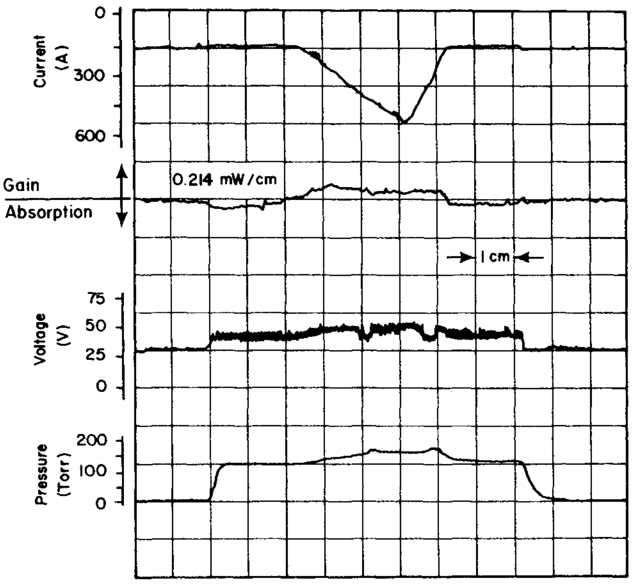

Fig. 1 Gain measurement at 5145Å, $\dot{m} = 4.8$ g/sec, $P_0 = 130$ torr, chart speed = 0.25 cm/sec, correlation trace.

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Method for Determining the Effect of Added Stores on Aeroelastic Systems

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In this Note, a method is presented for determining the effect of an added store on the stability of a given aeroelastic system. It is assumed that the addition of the store affects only the mass characteristics of the system, with the aerodynamic and elastic properties remaining unchanged. The mass of the added store, together with its c.g. and radius of gyration, is treated as an auxiliary variable in the flutter equation. The

Received July 24, 1973; revision received July 1, 1974.
Index category: Aeroelasticity.

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result is a plot of critical store mass vs c.g. location or store inertia, for a fixed flight velocity. This type of dependency was first considered by Epperson,[1] who obtained the results with the aid of an electronic analog. The present approach permits similar results to be obtained on a digital computer.

One special feature of the assumed velocity concept is that it permits one to evaluate the aerodynamics in agreement with the selected velocity, rather than for an assumed Mach number which cannot be predicted in advance when classical flutter analyses are made.

Mass Dependency Method

The aeroelastic stability equation for a system with n bending and n torsional degrees of freedom may be written in terms of a mass, stiffness, and aerodynamic matrix as follows:

$$\begin{pmatrix} h \\ \theta \end{pmatrix} = \omega^2 \begin{bmatrix} H & \\ & \theta \end{bmatrix} \begin{bmatrix} m+\tilde{L}_h & S+\tilde{L}_\theta \\ S+\tilde{M}_h & I+\tilde{M}_\theta \end{bmatrix} \begin{pmatrix} h \\ \theta \end{pmatrix} \tag{1}$$

where h and θ are bending and torsional displacements, H and θ the matrices of bending and torsion influence coefficients, m the mass matrix, S the static unbalance matrix, and I the moment of inertia matrix. $\tilde{L}_h, \tilde{L}_\theta, \tilde{M}_h, \tilde{M}_\theta$ aerodynamic matrices describing the lift and moment about the elastic axis, and ω is the circular frequency. The aerodynamic matrices are functions of the reduced frequency, $k = b\omega/v$, where v is the forward velocity and b is a reference length (usually the semichord). Equation (1) may be written in the form

$$[I - \omega^2 A^*]Y = 0 \tag{2}$$

where Y is the $(n \times 1)$ column matrix $\binom{h}{\theta}$.

Equation (2) is conventionally handled by equating to zero the determinant of the system

$$|I - \omega^2 A^*| = 0 \tag{3}$$

which must then be solved for several values of the reduced frequency in order to find a value where ω is real. The location of this critical value of k is usually accomplished graphically.

An alternative approach which also allows the effect of an added mass to be included is the following. Let A be the matrix $[I/\omega^2 - A^*]$ evaluated for some fixed values of ω^2 and k (or, equivalently, for a fixed value of v and k). If the selected values of k and v are not the critical ones, $|A| \neq 0$. Assume now that a mass M with c.g. location x and radius of gyration r is added at some point. Then matrix $[A]$ will be replaced by the following matrix:

$$[A] + M[B] + Mx[C+D] + M(x^2+r^2)[E] \tag{4}$$

In the preceding expression, the matrices B, C, D, and E are $(2n)$ column matrices whose elements are either zero or equal to the values of the influence coefficients at the station where the mass is added. Specifically, if the mass is added at station "j," then

$$B = \begin{bmatrix} H_{1j} \\ H_{2j} \\ \vdots \\ H_{nj} \\ \hline 0 \end{bmatrix} \qquad C = \begin{bmatrix} H_{1j} \\ H_{2j} \\ \vdots \\ H_{nj} \\ \hline 0 \end{bmatrix}$$

(jth column) $(n+j$th column)

$$D = \begin{bmatrix} 0 \\ \hline \theta_{1j} \\ \theta_{2j} \\ \vdots \\ \theta_{nj} \end{bmatrix} \qquad E = \begin{bmatrix} 0 \\ \hline \theta_{1j} \\ \theta_{2j} \\ \vdots \\ \theta_{nj} \end{bmatrix} \tag{5}$$

(jth column) $(n+j$th column)

By expanding the determinant of Eq. (4), the frequency equation may be written in the form

$$(A_1+iA_2)+(B_1+iB_2)M+(C_1+iC_2)Mx+ (D_1+iD_2)M(x^2+r^2)+(E_1+iE_2)M^2r^2 = 0 \tag{6}$$

In Eq. (6), A_1+iA_2 is the determinant of the original matrix, while the remaining quantities are the determinants obtained from A when the appropriate columns are replaced by one of the matrices B, C, D, or E. Specifically, B_1+iB_2 is the determinant obtained after the jth column A has been replaced by B, while D_1+iD_2 is obtained in a similar way by replacing the $(n+j)$th column of A by the elements of E. C_1+iC_2, on the other hand, is the sum of two determinants, one obtained by replacing the jth column of A by D and the other by replacing the $(n+j)$th column by C. Finally, to get E_1+iE_2 we replace the jth column of A by B and the $(n+j)$th column by E.

Equation (7) is equivalent to the two equations

$$\begin{aligned} A_1+B_1M+C_1Mx+D_1M(x^2+r^2)+E_1M^2r^2 &= 0 \\ A_2+B_2M+C_2Mx+D_2M(x^2+r^2)+E_2M^2r^2 &= 0 \end{aligned} \tag{7}$$

in the three variables M, x, and r. It follows that, for a fixed value of any one of these quantities, pairs of values for the other two may be determined which satisfy Eqs. (7). This leads to the following cases:

Case I: r fixed

Equations (7) become

$$\begin{aligned} A_1+\bar{B}_1M+C_1Mx+D_1Mx^2+\bar{E}_1M^2 &= 0 \\ A_2+\bar{B}_2M+C_2Mx+D_2Mx^2+\bar{E}_2M^2 &= 0 \end{aligned} \tag{8}$$

where $\bar{B}_1 = B_1+D_1r^2$, $\bar{B}_2 = B_2+D_2r^2$, $\bar{E}_1 = r^2E_1$, and $\bar{E}_2 = r^2E_2$.

Eliminating x gives a quartic equation in M

$$R_1M^4+(2R_1Q_1-\alpha R_2)M^3+(2R_1P_1+Q_1{}^2-\alpha Q_2)M^2+ (2P_1Q_1-\alpha P_2)M+P_1{}^2 = 0 \tag{9}$$

where

$$\begin{aligned} &P_1 = A_1D_2-A_2D_1,\ Q_1 = \bar{B}_1D_2-\bar{B}_2D_1,\ R_1 = \bar{E}_1D_2-\bar{E}_2D_1 \\ &P_2 = A_1C_2-A_2C_1,\ Q_2 = \bar{B}_1C_2-\bar{B}_2C_1,\ R_2 = \bar{E}_1C_2-\bar{E}_2C_1 \\ &\alpha = C_1D_2-C_2D_1 \end{aligned} \tag{10}$$

When M has been determined, the corresponding values of x can be found from either of the expressions

$$x = -(P_1+Q_1M+R_1M^2)/\alpha M$$

or

$$x = -(P_2+Q_2M+R_2M^2)/(P_1+Q_1M+R_1M^2) \tag{11}$$

Case II: x fixed

Here, Eqs. (7) reduce to

$$\begin{aligned} A_1+\bar{B}_1M+D_1Mr^2+E_1M^2r^2 &= 0 \\ A_2+\bar{B}_2M+D_2Mr^2+E_2M^2r^2 &= 0 \end{aligned} \tag{12}$$

with

$$\begin{aligned} \bar{B}_1 &= B_1+C_1x+D_1x^2 \\ \bar{B}_2 &= B_2+C_2x+D_2x^2 \end{aligned} \tag{13}$$

Thus

$$\begin{aligned} &[\bar{B}_1E_2-\bar{B}_2E_1]M^2+[(\bar{B}_1D_2-\bar{B}_2D_1)+ (A_1E_2-A_2E_1)]M+[A_1D_2-A_2D_1] = 0 \\ &r^2 = -(A_1+\bar{B}_1M)/M(D_1+E_1M) = -(A_2+\bar{B}_2M)/M(D_2+E_2M) \end{aligned} \tag{14}$$

Case III: M fixed

Equations (7) take the form

$$\begin{aligned} \bar{A}_1+\bar{C}_1x+\bar{D}_1(x^2+r^2) &= 0 \\ \bar{A}_2+\bar{C}_2x+\bar{D}_2(x^2+r^2) &= 0 \end{aligned} \tag{15}$$

$$\begin{aligned} \bar{A}_1 &= M(A_1+B_1) & \bar{A}_2 &= M(A_2+B_2) \\ \bar{C}_1 &= M(C_1) & \bar{C}_2 &= M(C_2) \\ \bar{D}_1 &= M(D_1) & \bar{D}_2 &= M(D_2) \\ \bar{E}_1 &= M(D_1+E_1M) & \bar{E}_2 &= M(D_2+E_2M) \end{aligned} \tag{16}$$

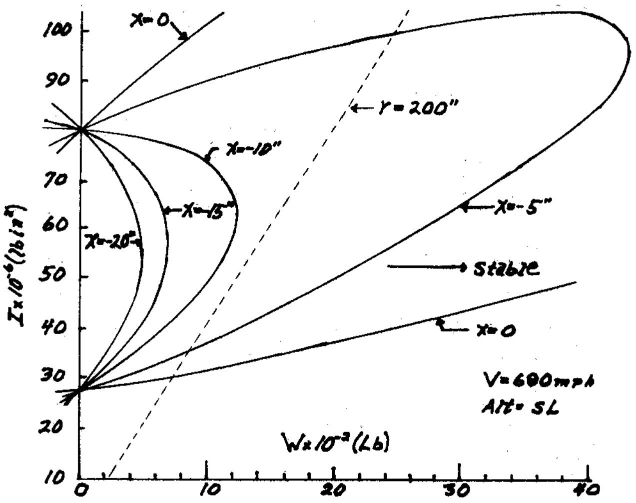

Fig. 1 Variation of critical values of store weight and store inertia, at an assumed airspeed of 680 mph, for several c.g. locations, x.

Hence

$$(\bar{D}_1 \bar{E}_2 - \bar{D}_2 \bar{E}_1)x^2 + (\bar{C}_1 \bar{E}_2 - \bar{C}_2 \bar{E}_1)x + (\bar{A}_1 \bar{E}_2 - \bar{A}_2 \bar{E}_1) = 0 \quad (17)$$

$$r^2 = -\bar{A}_1 + \bar{C}_1 x + \bar{D}_1 x^2/\bar{E}_1 = -\bar{A}_2 + \bar{C}_2 x + \bar{D}_2 x^2/\bar{E}_2 \quad (18)$$

Numerical Example

Figure 1 shows the results obtained for a typical system[2] with an assumed velocity of 680 mph and consists of plots of store weight W vs store inertia $I = Wr^2$ for five different c.g. locations x. Since the selected velocity in this case lies below the critical flutter speed of the system, the origin of the plot is in the stable region, i.e., in the portion of the plane exterior to all of the contours. Had the assumed velocity been equal to the critical one, the contours would have all passed through the origin, while for velocities above the critical one, the origin would lie interior to all of the curves. As would be anticipated, the region of instability becomes larger as the c.g. location is moved aft.

By assuming a range of velocities, and considering a store with a fixed radius of gyration and c.g. location, the plots of store weight vs store inertia may be converted into a single plot of flutter speed vs store weight. This is accomplished by determining the intersections of the appropriate weight-vs-inertia contour with the straight line; $I = Wr^2$ for the various assumed values of velocity. Coordinates of the desired plot of critical velocity vs store weight are then easily found by cross plotting.

Comparison with Other Methods in Current Use

Some of the more frequently used approaches to the present problem employ the perturbation method[3,4] which shows only the effect of small changes in the store mass and, in addition, usually requires a complete eigenvalue-eigenvector analysis of the flutter matrix, itself a time-consuming process. In contrast, the present method allows one to investigate a wide range of store loadings and unbalances by means of a relatively simple program involving nothing more elaborate than determinant evaluation and the solution of an algebraic equation with real coefficients. The input consists simply of basic mass, stiffness, and aerodynamic data for the wing structure itself, together with a range of velocities and store c.g. locations with output consisting of critical values of store mass and store inertia. Such outputs may be either numerical or graphical if a suitable plot routine is available. Further, the method does not require the use of complicated and specialized electronic analogs as does the approach described in Ref. 1.

In conclusion, it may be added that the method may be adapted to modal-type flutter analyses. In this case the addition of a store results in an augmented matrix of the following form:

$$\begin{bmatrix} A_{i,j} + Mh_i h_j & A_{i,n+j} + Mxh_i\theta_j \\ A_{n+1,j} + Mxh_i\theta_j & A_{n+i,n+j} + M(x^2+r^2)\theta_i\theta_j \end{bmatrix} \quad (19)$$

where h_i and θ_j are the deflections of the various modes at the station where the mass has been added. By multiplying column 1 by (h_j/h_1), column $(n+1)$ by (θ_j/θ_1) and subtracting, respectively, from columns 2 through n and $(n+2)$ through $2n$, the matrix is brought into the same form as Eq. (4), with the additional terms appearing only in the first and $(n+1)$st column.

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Integral Approximation for Slender-Body Shock Shapes in Hypersonic Flow

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I. Introduction

THERE have been a number of different methods devised to calculate the shock shapes and flowfields associated with hypersonic flows past slender bodies. Various references germane to the present Note are discussed in Ref. 1. Depending on the objectives desired, the calculation methods soon evolve into lengthy numerical schemes or involved analytical schemes that focus on accuracy and certain specific details, as opposed to consideration of over-all general features of flow problems. In this Note we concern ourselves with a relatively simple method due to Chernyi.[2] By means of integral approximations, Chernyi arrives at a pair of ordinary differential equations that describe the shock shapes and body pressure distribution associated with slender planar bodies and bodies of revolution. In this Note we show that the pair of equations reduces to a single quadrature for the body shape when the shock shape is known. The pressure distribution is determined entirely in terms of the

Received November 15, 1973; revision received July 2, 1974.
Index categories: Shock Waves and Detonations; Supersonic and Hypersonic Flow.

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